

Lipschitz stability for the inverse conductivity problem for a conformal class of anisotropic conductivities

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Abstract. We consider the stability issue of the inverse conductivity problem for a conformal class of anisotropic conductivities in terms of the local Dirichlet-to-Neumann map. We extend here the stability result obtained by Alessandrini and Vessella in *Advances in Applied Mathematics* 35:207-241, where the authors considered the piecewise constant isotropic case.

1 Introduction

In the present paper we study the stability issue for the inverse conductivity problem in the presence of anisotropic conductivity which is *a-priori* known to depend linearly on a unknown piecewise-constant function. Let us start by recalling the basic formulation of the inverse conductivity problem.

In absence of internal sources, the electrostatic potential u in a conducting body, described by a domain $\Omega \subset \mathbb{R}^n$, is governed by the elliptic equation

$$\operatorname{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega, \quad (1.1)$$

where the symmetric, positive definite matrix $\sigma = \sigma(x)$, $x \in \Omega$ represents the (possibly anisotropic) electric conductivity. The inverse conductivity problem consists of finding σ when the so called Dirichlet-to-Neumann (D-N) map

$$\Lambda_\sigma : u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega) \longrightarrow \sigma \nabla u \cdot \nu|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$$

is given for any $u \in H^1(\Omega)$ solution to (1.1). Here, ν denotes the unit outer normal to $\partial\Omega$. If measurements can be taken only on one portion Σ of $\partial\Omega$, then the relevant map is called the local Dirichlet-to-Neumann map. Let Σ be a non-empty open portion of $\partial\Omega$ and let us introduce the subspace of $H^{\frac{1}{2}}(\partial\Omega)$

$$H_{co}^{\frac{1}{2}}(\Sigma) = \{f \in H^{\frac{1}{2}}(\partial\Omega) \mid \operatorname{supp} f \subset \Sigma\}. \quad (1.2)$$

The local D-N map is given, in its weak formulation, as the operator Λ_σ^Σ such that

$$\langle \Lambda_\sigma^\Sigma u, \phi \rangle = \int_\Omega \sigma \nabla u \cdot \nabla \phi, \quad (1.3)$$

for any $u, \phi \in H^1(\Omega)$, $u|_{\partial\Omega}, \phi|_{\partial\Omega} \in H_{co}^{\frac{1}{2}}(\Sigma)$ and u is a weak solution to (1.1).

The problem of recovering the conductivity of a body by taking measurements of voltage and current on its surface has come to be known as Electrical Impedance Tomography (EIT). Different materials display different electrical properties, so that a map of the conductivity $\sigma(x)$, $x \in \Omega$ can be used to investigate internal properties of Ω . EIT has many important applications in fields such as geophysics, medicine and non-destructive testing of materials. The first mathematical formulation of the inverse conductivity problem is due to A. P. Calderón [C], where he addressed the problem of whether it is possible to determine the (isotropic) conductivity $\sigma = \gamma I$ by the D-N map. Although Calderón studied the problem of determining σ from the knowledge of the quadratic form

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$$Q_\gamma(u) = \int_{\Omega} \gamma |\nabla u|^2,$$

where u is a solution to (1.1), it is well known that the knowledge of Q_σ is equivalent to the knowledge of Λ_σ by

$$Q_\gamma(u) = \langle \Lambda_\sigma u, u \rangle, \quad \text{for every } u \in H^1(\Omega),$$

where $\sigma = \gamma I$. Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $H^{1/2}(\partial\Omega)$ and its dual $H^{-1/2}(\partial\Omega)$, with respect to the L^2 scalar product. [C] opened the way to the solution to the uniqueness issue where one is asking whether σ can be determined by the knowledge of Λ_σ (or Λ_σ^Σ in the case of local measurements). As main contributions in this respect we mention the papers by Kohn and Vogelius [K-Vo1, K-Vo2], Sylvester and Uhlmann [S-U] and Nachman [Na]. We refer to [Bo], [Ch-I-N] and [U] for an overview of recent developments regarding the issues of uniqueness and reconstruction of the conductivity.

Regarding the stability, Alessandrini proved in [A] that, assuming $n \geq 3$ and *a-priori* bounds on γ of the form

$$\|\gamma\|_{H^s(\Omega)} \leq E, \quad \text{for some } s > \frac{n}{2} + 2, \quad (1.4)$$

γ depends continuously on Λ_σ with a modulus of continuity of logarithmic type. In [A1], [A2] the same author subsequently proved that a similar stability estimate holds when the *a-priori* bound (1.4) is replaced by

$$\|\gamma\|_{W^{2,\infty}(\Omega)} \leq E. \quad (1.5)$$

For the two-dimensional case, logarithmic type stability estimates were obtained in [B-B-R], [B-F-R] and [Liu]. Unfortunately, all the above results have the common inconvenient logarithmic type of stability which cannot be avoided [A3]. In fact Mandache [Ma] showed that the logarithmic stability is the best possible, in any dimension $n \geq 2$ if *a-priori* assumptions of the form

$$\|\gamma\|_{C^k(\Omega)} \leq E \quad (1.6)$$

for any $k = 0, 1, 2, \dots$ are assumed. It seems therefore reasonable to think that, in order to restore stability in a really (Lipschitz) stable fashion, one needs to replace in some way the *a-priori* assumptions expressed in terms of regularity bounds such as (1.6), with *a-priori* pieces of information of a different type. Alessandrini and Vessella showed in [A-V] that γ depends in a Lipschitz continuous fashion upon the local D-N map, by assuming that γ is a function *a-priori* known to be piecewise constant

$$\gamma(x) = \sum_{j=1}^N \gamma_j \chi_{D_j}(x), \quad (1.7)$$

where each subdomain of Ω , D_j , $j = 1, \dots, N$ is *given* and each number γ_j , $j = 1, \dots, N$ is *unknown*. From a medical imaging point of view, each D_j may represent the area occupied by different tissues or organs and one can think that the geometrical configuration of each D_j is given by means of other imaging techniques such as MRI for example. Since most tissues in the human body are anisotropic, the present authors, motivated by the work in [A-V] and its medical application, consider here the more general case of an *anisotropic* conductivity of type

$$\sigma(x) = \gamma(x)A(x),$$

where $A(x)$ is a known, matrix valued function which is Lipschitz continuous and $\gamma(x)$ is of type (1.7). The authors would like to stress out that anisotropic conductivity appears in nature, for example as a

homogenization limit in layered or fibrous structures such as rock stratum or muscle, as a result of crystalline structure or of deformation of an isotropic material, therefore the case treated in this paper seems to be a natural extension of [A-V] relevant to several applications. For related results in the anisotropic case we also refer to [A-G], [A-G1], [A-L-P], [Be], [F-K-R], [G-L], [L] and [La-U]). The present paper improves upon the results obtained in [A-V] in the sense that the global Lipschitz stability estimate obtained there is here adapted to a special anisotropic type of conductivity. The precise assumptions shall be illustrated in section 2. We also recall [Be-Fr], [Be-Fr-V] and [Be-dH-Q] where similar Lipschitz stability results have been obtained for complex conductivity, the Lamé parameters and for a Schrödinger type of equation respectively. For a more in-dept description and consideration of the stability issue and related open problems in the inverse conductivity problem we refer to [A3] and [A-V].

Our approach follows the one by Alessandrini and Vessella [A-V] of constructing singular solutions and studying their asymptotic behaviour when the singularity approaches the discontinuity interfaces. However, in order to deal with the present structure of conductivity we had to develop original asymptotic analysis estimates and an accurate quantitative control of the error terms which represent a novel feature in the treatment of anisotropic type of conductivity.

The paper is organized as follows. Our main assumptions and our main result (Theorem 2.1) are contained in section 2, where the proof of Theorem 2.1 is contained in section 3. This section also lists the two main results (Theorem 3.4 and Proposition 3.5) needed to build the machinery for the proof of Theorem 2.1. Theorem 3.4 provides original asymptotic estimates for the Green function of the conductivity equation, for conductivities belonging to a special anisotropic conformal class \mathcal{C} , at the interfaces between the given domains D_j , where the conductivity is discontinuous. Proposition 3.5 provides estimates of unique continuation of the solution to the conductivity equation for conductivities in \mathcal{C} . Section 4 is devoted to the proof of Theorem 3.4 and Proposition 3.5. For the proof of Theorem 3.4 we provide the explicit form of the fundamental solution for the conformal anisotropic two-phase case with flat interface. The proof of Proposition 3.5 is a straight forward consequence Proposition 4.3 which we state in this section. The proof of the latter is independent from the presence of anisotropy in the conductivity, therefore we refer to [A-V] for a full proof of it. In this paper we point out the main facts on which the proof is based on only.

2 Main Result

2.1 Notation and definitions

In several places within this manuscript it will be useful to single out one coordinate direction. To this purpose, the following notations for points $x \in \mathbb{R}^n$ will be adopted. For $n \geq 3$, a point $x \in \mathbb{R}^n$ will be denoted by $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Moreover, given a point $x \in \mathbb{R}^n$, we shall denote with $B_r(x), B'_r(x)$ the open balls in $\mathbb{R}^n, \mathbb{R}^{n-1}$ respectively centred at x with radius r and by $Q_r(x)$ the cylinder

$$Q_r(x) = B'_r(x') \times (x_n - r, x_n + r).$$

We shall also denote

$$\begin{aligned} \mathbb{R}_+^n &= \{(x', x_n) \in \mathbb{R}^n | x_n > 0\}; \quad \mathbb{R}_-^n = \{(x', x_n) \in \mathbb{R}^n | x_n < 0\}; \\ B_r^+ &= B_r \cap \mathbb{R}_+^n; \quad B_r^- = B_r \cap \mathbb{R}_-^n; \\ Q_r^+ &= Q_r \cap \mathbb{R}_+^n; \quad Q_r^- = Q_r \cap \mathbb{R}_-^n. \end{aligned}$$

In the sequel, we shall make a repeated use of quantitative notions of smoothness for the boundaries of various domains. Let us introduce the following notation and definitions.

DEFINITION 2.1. Let Ω be a domain in \mathbb{R}^n . We say that a portion Σ of $\partial\Omega$ is of Lipschitz class with constants r_0, L if for any $P \in \partial\Sigma$ there exists a rigid transformation of \mathbb{R}^n under which we have $P \equiv 0$ and

$$\Omega \cap Q_{r_0} = \{x \in Q_{r_0} : x_n > \varphi(x')\},$$

where φ is a Lipschitz function on B'_{r_0} satisfying

$$\varphi(0) = |\nabla_{x'} \varphi(0)| = 0; \quad \|\varphi\|_{C^{0,1}(B'_{r_0})} \leq Lr_0.$$

It is understood that $\partial\Omega$ is of Lipschitz class with constants r_0, L as a special case of Σ , with $\Sigma = \partial\Omega$.

DEFINITION 2.2. Let Ω be a domain in \mathbb{R}^n . Given $\alpha, \alpha \in (0, 1]$, we say that a portion Σ of $\partial\Omega$ is of class $C^{1,\alpha}$ with constants r_0, M if for any $P \in \Sigma$ there exists a rigid transformation of \mathbb{R}^n under which we have $P = 0$ and

$$\Omega \cap Q_{r_0} = \{x \in Q_{r_0} : x_n > \varphi(x')\},$$

where φ is a $C^{1,\alpha}$ function on B'_{r_0} satisfying

$$\varphi(0) = |\nabla_{x'} \varphi(0)| = 0; \quad \|\varphi\|_{C^{1,\alpha}(B'_{r_0})} \leq Mr_0,$$

where we denote

$$\|\varphi\|_{C^{1,\alpha}(B'_{r_0})} = \|\varphi\|_{L^\infty(B'_{r_0})} + r_0 \|\nabla \varphi\|_{L^\infty(B'_{r_0})} + r_0^{1+\alpha} \sup_{\substack{x, y \in B'_{r_0} \\ x \neq y}} \frac{|\nabla \varphi(x) - \nabla \varphi(y)|}{|x - y|^\alpha}.$$

Let us rigorously define the local D-N map.

DEFINITION 2.3. Let Ω be a domain in \mathbb{R}^n with Lipschitz boundary $\partial\Omega$ and Σ an open non-empty subset of $\partial\Omega$. Assume that $\sigma \in L^\infty(\Omega, \text{Sym}_n)$ satisfies the ellipticity condition

$$\begin{aligned} \lambda^{-1} |\xi|^2 \leq \sigma(x) \xi \cdot \xi \leq \lambda |\xi|^2, \quad & \text{for almost every } x \in \Omega, \\ & \text{for every } \xi \in \mathbb{R}^n. \end{aligned} \tag{2.1}$$

The local Dirichlet-to-Neumann map associated to σ and Σ is the operator

$$\Lambda_\sigma^\Sigma : H_{co}^{\frac{1}{2}}(\Sigma) \longrightarrow H_{co}^{-\frac{1}{2}}(\Sigma) \tag{2.2}$$

defined by

$$\langle \Lambda_\sigma^\Sigma g, \eta \rangle = \int_\Omega \sigma(x) \nabla u(x) \cdot \nabla \phi(x) \, dx, \tag{2.3}$$

for any $g, \eta \in H_{co}^{\frac{1}{2}}(\Sigma)$, where $u \in H^1(\Omega)$ is the weak solution to

$$\begin{cases} \operatorname{div}(\sigma(x) \nabla u(x)) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases}$$

and $\phi \in H^1(\Omega)$ is any function such that $\phi|_{\partial\Omega} = \eta$ in the trace sense. Here we denote by $\langle \cdot, \cdot \rangle$ the $L^2(\partial\Omega)$ -pairing between $H_{co}^{\frac{1}{2}}(\Sigma)$ and its dual $H_{co}^{-\frac{1}{2}}(\Sigma)$.

Note that, by (2.3), it is easily verified that Λ_σ^Σ is selfadjoint. We shall denote by $\|\cdot\|_*$ the norm on the Banach space of bounded linear operators between $H_{co}^{\frac{1}{2}}(\Sigma)$ and $H_{co}^{-\frac{1}{2}}(\Sigma)$.

2.2 Our assumptions

We give here the precise assumptions for the domain Ω under investigation and its conductivity σ . The dimension of the space for Ω is denoted by n and for sake of simplicity is only consider $n \geq 3$.

2.2.1 Assumptions about the domain Ω

1. We assume that Ω is a domain in \mathbb{R}^n satisfying

$$|\Omega| \leq N r_0^n, \quad (2.4)$$

where $|\Omega|$ denotes the Lebesgue measure of Ω .

2. We assume that $\partial\Omega$ is of Lipschitz class with constants r_0, L .
3. We fix an open non-empty subset Σ of $\partial\Omega$ (where the measurements in terms of the local D-N map are taken).
- 4.

$$\bar{\Omega} = \bigcup_{j=1}^N \bar{D}_j,$$

where $D_j, j = 1, \dots, N$ are known open sets of \mathbb{R}^n , satisfying the conditions below.

- (a) $D_j, j = 1, \dots, N$ are connected and pairwise nonoverlapping.
- (b) $\partial D_j, j = 1, \dots, N$ are of Lipschitz class with constants r_0, L .
- (c) There exists one region, say D_1 , such that $\partial D_1 \cap \Sigma$ contains a $C^{1,\alpha}$ portion Σ_1 with constants r_0, M .
- (d) For every $i \in \{2, \dots, N\}$ there exists $j_1, \dots, j_K \in \{1, \dots, N\}$ such that

$$D_{j_1} = D_1, \quad D_{j_K} = D_i. \quad (2.5)$$

In addition we assume that, for every $k = 1, \dots, K$, $\partial D_{j_k} \cap \partial D_{j_{k-1}}$ contains a $C^{1,\alpha}$ portion Σ_k (here we agree that $D_{j_0} = \mathbb{R}^n \setminus \Omega$), such that

$$\Sigma_1 \subset \Sigma,$$

$$\Sigma_k \subset \Omega, \quad \text{for every } k = 2, \dots, K,$$

and, for every $k = 1, \dots, K$, there exists $P_k \in \Sigma_k$ and a rigid transformation of coordinates under which we have $P_k = 0$ and

$$\begin{aligned} \Sigma_k \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n = \phi_k(x')\} \\ D_{j_k} \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n > \phi_k(x')\} \\ D_{j_{k-1}} \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n < \phi_k(x')\}, \end{aligned} \quad (2.6)$$

where ϕ_k is a $C^{1,\alpha}$ function on $B'_{r_0/3}$ satisfying

$$\phi_k(0) = |\nabla \phi_k(0)| = 0$$

and

$$\|\phi_k\|_{C^{1,\alpha}(B'_{r_0})} \leq Mr_0.$$

2.2.2 A-priori information on the conductivity γ : the class \mathcal{C}

DEFINITION 2.4. We shall say that $\sigma \in \mathcal{C}$ if σ is of type

$$\sigma_A(x) = \sum_{j=1}^N \gamma_j A(x) \chi_{D_j}(x), \quad x \in \Omega, \quad (2.7)$$

where γ_j are unknown real numbers, D_j , $j = 1, \dots, N$ are the given subdomains introduced in section 2.2.1 and

$$\bar{\gamma} \leq \gamma_j \leq \bar{\gamma}^{-1}, \quad \text{for any } j = 1, \dots, n. \quad (2.8)$$

$A(x)$ is a known Lipschitz matrix valued function satisfying

$$\|A\|_{C^{0,1}(\Omega)} \leq \bar{A}, \quad (2.9)$$

where $\bar{A} > 0$ is a constant and

$$\lambda^{-1}|\xi|^2 \leq A(x)\xi \cdot \xi \leq \lambda|\xi|^2, \quad \text{for almost every } x \in \Omega, \quad (2.10)$$

for every $\xi \in \mathbb{R}^n$.

DEFINITION 2.5. Let $N, r_0, L, M, \alpha, \lambda, \bar{\gamma}, \bar{A}$ be given positive numbers with $N \in \mathbb{N}$ and $\alpha \in (0, 1]$. We will refer to this set of numbers, along with the space dimension n , as to the a-priori data.

THEOREM 2.1. Let $\Omega, D_j, j = 1, \dots, N$ and Σ be a domain, N subdomains of Ω and a portion of $\partial\Omega$ as in section 2.2.1 respectively. If $\sigma_A^{(i)} \in \mathcal{C}$, $i = 1, 2$ are two conductivities of type

$$\sigma_A^{(i)}(x) = \sum_{j=1}^N \gamma_j^{(i)} A(x) \chi_{D_j}(x) \quad x \in \Omega, \quad i = 1, 2, \quad (2.11)$$

then we have

$$\|\sigma_A^{(1)} - \sigma_A^{(2)}\|_{L^\infty(\Omega)} \leq C \|\Lambda_{\sigma_A^{(1)}}^\Sigma - \Lambda_{\sigma_A^{(2)}}^\Sigma\|_*, \quad (2.12)$$

where C is a positive constant that depends on the a-priori data only.

3 Proof of the main result

The proof of our main result (theorem 2.1) is based on an argument that combines asymptotic type of estimates for the Green's function of the operator

$$L = \operatorname{div}(\sigma(x)\nabla) \quad \text{in } \Omega, \quad (3.1)$$

(theorem 3.4), with $\sigma \in \mathcal{C}$, together with a result of unique continuation (proposition 3.5) for solutions to

$$Lu = 0, \quad \text{in } \Omega.$$

We shall give the precise formulation of these results in what follows.

3.1 Measurable conductivity σ

We shall start with some general considerations about the Green's function $G(x, y)$ associated to the operator (3.1), where σ is merely a measurable matrix valued function satisfying the ellipticity condition (2.1).

3.1.1 Green's function

If L is the operator given in (3.1), then for every $y \in \Omega$, the Green's function $G(\cdot, y)$ is the weak solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma \nabla G(\cdot, y)) = -\delta(\cdot - y), & \text{in } \Omega, \\ G(\cdot, y) = 0, & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

where $\delta(\cdot - y)$ is the Dirac measure at y . We recall that G satisfies the properties ([Lit-St-W])

$$G(x, y) = G(y, x) \quad \text{for every } x, y \in \Omega, \quad x \neq y, \quad (3.2)$$

$$0 < G(x, y) < |x - y|^{2-n} \quad \text{for every } x, y \in \Omega, \quad x \neq y. \quad (3.3)$$

Moreover, the following result holds true.

Proposition 3.1. *For any $y \in \Omega$ and every $r > 0$ we have that*

$$\int_{\Omega \setminus B_r(y)} |\nabla G(\cdot, y)|^2 \leq Cr^{2-n} \quad (3.4)$$

where $C > 0$ depends on λ and n only.

Proof. The proof can be obtained by combining Caccioppoli inequality with (3.3) ([A-V], Proposition 3.1). \square

3.1.2 Integral solutions of L

Let $\sigma^{(i)}$, $i = 1, 2$ be two measurable matrix valued functions satisfying the ellipticity condition (2.1) and let $G_i(x, y)$ be the Green's functions associated to the operators

$$L_i = \operatorname{div}(\sigma^{(i)}(x) \nabla) \quad \text{in } \Omega, \quad i = 1, 2. \quad (3.5)$$

Let \mathcal{U} be an open subset of Ω and $\mathcal{W} = \Omega \setminus \overline{\mathcal{U}}$. For any $y, z \in \mathcal{W}$ we define

$$S_{\mathcal{U}}(y, z) = \int_{\mathcal{U}} (\sigma^{(1)}(x) - \sigma^{(2)}(x)) \nabla_x G_1(x, y) \cdot \nabla_x G_2(z, x) dx. \quad (3.6)$$

Remark 3.2.

$$|S_{\mathcal{U}}(y, z)| \leq C \|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)} (d(y)d(z))^{1-\frac{n}{2}}, \quad \text{for every } y, z \in \mathcal{W}, \quad (3.7)$$

where $d(y) = \operatorname{dist}(y, \mathcal{U})$ and C is a positive constant depending on $\lambda, \bar{\gamma}$ and n only.

Observe that (3.7) is a straightforward consequence of Hölder inequality and Proposition 3.1. We constructed in this way an integral function $S_{\mathcal{U}}(\cdot, \cdot)$ on $\mathcal{W} \times \mathcal{W}$, which is written in terms of the two Green's functions $G_1(\cdot, y)$, $G_2(\cdot, z)$ of L_1 , L_2 respectively; $S_{\mathcal{U}}(\cdot, z)$, $S_{\mathcal{U}}(y, \cdot)$ are in turn solutions for L_1 , L_2 respectively on the complement part of \mathcal{U} in Ω . More precisely we have

THEOREM 3.3. For every $y, z \in W$ we have that $S_{\mathcal{U}}(\cdot, z), S_{\mathcal{U}}(y, \cdot) \in H_{loc}^1(W)$ are weak solutions to

$$\operatorname{div} \left(\sigma^{(1)}(\cdot) \nabla S_{\mathcal{U}}(\cdot, z) \right) = 0, \quad \operatorname{div} \left(\sigma^{(2)}(\cdot) \nabla S_{\mathcal{U}}(y, \cdot) \right) = 0, \quad \text{in } \mathcal{W}. \quad (3.8)$$

Proof. The proof relies on differentiation under the integral sign arguments and the symmetry of $G_i, i = 1, 2$. \square

3.2 Conductivity $\sigma \in \mathcal{C}$

We shall denote with

$$\Gamma(x, y) = \frac{1}{(n-2)\omega_n} |x - y|^{2-n}, \quad (3.9)$$

the fundamental solution of the Laplace operator (here ω_n/n denotes the volume of the unit ball in \mathbb{R}^n). If $D_i, i = 1, \dots, N$ are the domains introduced in section 2.2.1 and L is the operator given by (3.1), with $\sigma \in \mathcal{C}$, we shall give asymptotic estimates for the Green's function of L , with respect to (3.9) at the interfaces between the domains $D_i, i = 1, \dots, N$. These estimates are given below. In what follows let G be the Green's function associated to the operator L in Ω .

3.2.1 Green's function

THEOREM 3.4. (**Asymptotic estimates**) For every $l \in \{1, \dots, K-1\}$, let $\nu(P_{l+1})$ denote the unit exterior normal to $D_{j_{l+1}}$ at the point P_{l+1} . There exist constants $\beta \in (0, \alpha)$ and $\bar{C} > 1$ depending on $\bar{\gamma}, \lambda, M, \alpha$ and n only such that the following inequalities hold true for every $\bar{x} \in B_{\frac{r_0}{\bar{C}}}(P_{l+1}) \cap D_{j_{l+1}}$ and every $\bar{y} = P_{l+1} + r\nu(P_{l+1})$, where $r \in (0, \frac{r_0}{\bar{C}})$

$$\left| G(\bar{x}, \bar{y}) - \frac{2}{\gamma_{j_l} + \gamma_{j_{l+1}}} \Gamma(J(\bar{x}), J(\bar{y})) \right| \leq \frac{\bar{C}}{r_0^\beta} |\bar{x} - \bar{y}|^{\beta+2-n}, \quad (3.10)$$

$$\left| \nabla_x G(\bar{x}, \bar{y}) - \frac{2}{\gamma_{j_l} + \gamma_{j_{l+1}}} \nabla_x \Gamma(J(\bar{x}), J(\bar{y})) \right| \leq \frac{\bar{C}}{r_0^\beta} |\bar{x} - \bar{y}|^{\beta+1-n}, \quad (3.11)$$

where J is the positive definite matrix such that $J = \sqrt{A(0)^{-1}}$.

3.2.2 Integral solutions of L : unique continuation

We recall that up to a rigid transformation of coordinates we can assume that

$$P_1 = 0 \quad ; \quad (\mathbb{R}^n \setminus \Omega) \cap B_{r_0} = \{(x', x_n) \in B_{r_0} \mid x_n < \varphi(x')\},$$

where φ is a Lipschitz function such that

$$\varphi(0) = 0 \quad \text{and} \quad \|\varphi\|_{C^{0,1}(B'_{r_0})} \leq Lr_0.$$

Denoting by

$$D_0 = \left\{ x \in (\mathbb{R}^n \setminus \Omega) \cap B_{r_0} \mid |x_i| < \frac{2}{3}r_0, \quad i = 1, \dots, n-1, \quad \left| x_n - \frac{r_0}{6} \right| < \frac{5}{6}r_0 \right\},$$

it turns out that the augmented domain $\Omega_0 = \Omega \cup D_0$ is of Lipschitz class with constants $\frac{r_0}{3}$ and \tilde{L} , where \tilde{L} depends on L only. We consider the operator L_i given by (3.5) and extend $\sigma^{(i)} \in \mathcal{C}$ to $\tilde{\sigma}^{(i)} = \tilde{\gamma}^{(i)} \tilde{A}$ on Ω_0 ,

by setting $\tilde{\gamma}^{(i)}|_{D_0} = 1$, and extending A to $\tilde{A} \in C^{0,1}(\Omega_0)$ with Lipschitz constant L , for $i = 1, 2$. We denote by \tilde{G}_i the Green function associated to $\tilde{L}_i = \operatorname{div}(\tilde{\sigma}^{(i)}(x)\nabla \cdot)$ in Ω_0 , for $i = 1, 2$. For any number $r \in (0, \frac{2}{3}r_0)$ we also denote

$$(D_0)_r = \{x \in D_0 \mid \operatorname{dist}(x, \Omega) > r\}.$$

Let us fix $k \in \{2, \dots, N\}$ and recall that there exist $j_1, \dots, j_K \in \{1, \dots, N\}$ such that

$$D_{j_1} = D_1, \dots, D_{j_K} = D_K.$$

We denote

$$\mathcal{W}_K = \bigcup_{i=0}^K D_{j_i}, \quad \mathcal{U}_k = \Omega_0 \setminus \overline{\mathcal{W}_K}, \quad \text{when } k \geq 0$$

$(D_{j_0} = D_0)$ and for any $y, z \in \mathcal{W}_K$

$$\tilde{S}_{\mathcal{U}_K}(y, z) = \int_{\mathcal{U}_K} (\tilde{\sigma}_A^{(1)} - \tilde{\sigma}_A^{(2)}) \nabla \tilde{G}_1(\cdot, y) \cdot \nabla \tilde{G}_2(\cdot, z), \quad \text{when } k \geq 0.$$

We introduce for any number $b > 0$ as in [A-V], the concave non decreasing function $\omega_b(t)$, defined on $(0, +\infty)$,

$$\omega_b(t) = \begin{cases} 2^b e^{-2} |\log t|^{-b}, & t \in (0, e^{-2}), \\ e^{-2}, & t \in [e^{-2}, +\infty) \end{cases}$$

and denote

$$\omega_b^{(1)} = \omega, \quad \omega_b^{(j)} = \omega_b \circ \omega_b^{(j-1)}.$$

The following parameters shall also be introduced

$$\begin{aligned} \beta &= \arctan \frac{1}{L}, \quad \beta_1 = \arctan \left(\frac{\sin \beta}{4} \right), \quad \lambda_1 = \frac{r_0}{1 + \sin \beta_1} \\ \rho_1 &= \lambda_1 \sin \beta_1, \quad a = \frac{1 - \sin \beta_1}{1 + \sin \beta_1} \\ \lambda_k &= a \lambda_{k-1}, \quad \rho_k = a \rho_{k-1}, \quad \text{for every } k \geq 2, \\ d_k &= \lambda_k - \rho_k, \quad k \geq 1. \end{aligned}$$

Let us denote here and in the sequel

$$E = \|\sigma_A^{(1)} - \sigma_A^{(2)}\|_{L^\infty(\Omega)}.$$

The following estimate for $\tilde{S}_{\mathcal{U}_K}(y, z)$ holds true.

Proposition 3.5. (Estimates of unique continuation) *If, for a positive number ε_0 , we have*

$$\left| \tilde{S}_{\mathcal{U}_K}(y, z) \right| \leq r_0^{2-n} \varepsilon_0, \quad \text{for every } (y, z) \in (D_0)_{\frac{r_0}{3}} \times (D_0)_{\frac{r_0}{3}}, \quad (3.12)$$

then the following inequality holds true for every $r \in (0, d_1]$

$$\left| \tilde{S}_{\mathcal{U}_K}(w_{\bar{h}}(P_{K+1}), w_{\bar{h}}(P_{K+1})) \right| \leq r_0^{2-n} C^{\bar{h}} (E + \varepsilon_0) \left(\omega_{1/C}^{(2K)} \left(\frac{\varepsilon_0}{E + \varepsilon_0} \right) \right)^{(1/C)^{\bar{h}}}, \quad (3.13)$$

where $P_{K+1} \in \Sigma_{K+1}$, $\bar{h} = \min\{k \in \mathbb{N} \mid d_k \leq r\}$, $w_{\bar{h}}(P_{K+1}) = P_{K+1} - \lambda_{\bar{h}} \nu(P_{K+1})$, ν is the exterior unit normal to ∂D_K and $C \geq 1$ depends on the a-priori data only.

3.3 Proof of Theorem 2.1

Proof. *Proof of Theorem 2.1.* We denote by Λ_i the map $\Lambda_{\sigma_A^{(i)}}^{(\Sigma)}$, for $i = 1, 2$ and, for every $k \in \{0, \dots, K\}$, the subscript j_k will be replaced by k . This should simplify the notation. Let us point out that

$$\|(\sigma_A^{(1)} - \sigma_A^{(2)})\|_{L^\infty(\Omega)} \leq \bar{A} \|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)},$$

where

$$\gamma^{(i)} = \sum_{j=1}^N \gamma_j^{(i)} \chi_{D_j}(x), \quad i = 1, 2,$$

therefore (2.12) trivially follows from

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)} \leq C \|\Lambda_1 - \Lambda_2\|_{\mathcal{L}(H_{co}^{1/2}(\Sigma), H_{co}^{-1/2}(\Sigma))} \quad (3.14)$$

which we shall prove. Moreover we shall denote

$$\varepsilon = \|\Lambda_1 - \Lambda_2\|, \quad \delta_k = \|\tilde{\gamma}^{(1)} - \tilde{\gamma}^{(2)}\|_{L^\infty(\mathcal{W}_k)}.$$

We start by recalling that for every $y, z \in D_0$ we have

$$\left\langle (\Lambda_1 - \Lambda_2) \tilde{G}_1(\cdot, y), \tilde{G}_2(\cdot, z) \right\rangle = \int_{\Omega} (\tilde{\gamma}^{(1)} - \tilde{\gamma}^{(2)}) A(\cdot) \nabla \tilde{G}_1(\cdot, y) \cdot \nabla \tilde{G}_2(\cdot, z)$$

and that, for every $k \in \{1, \dots, K\}$,

$$\tilde{S}_{\mathcal{U}_{k-1}}(y, z) = \int_{\mathcal{U}_{k-1}} (\tilde{\gamma}^{(1)} - \tilde{\gamma}^{(2)}) \tilde{A}(\cdot) \nabla \tilde{G}_1(\cdot, y) \cdot \nabla \tilde{G}_2(\cdot, z),$$

therefore

$$\begin{aligned} |\tilde{S}_{\mathcal{U}_{k-1}}(y, z)| &\leq \varepsilon \|\tilde{G}_1(\cdot, y)\|_{H_{co}^{1/2}(\Sigma)} \|\tilde{G}_2(\cdot, z)\|_{H_{co}^{1/2}(\Sigma)} \\ &\quad + \delta_{k-1} \bar{A} \|\nabla \tilde{G}_1(\cdot, y)\|_{L^2(\mathcal{W}_{k-1})} \|\nabla \tilde{G}_2(\cdot, z)\|_{L^2(\mathcal{W}_{k-1})} \\ &\leq C(\varepsilon + \delta_{k-1}) r_0^{2-n}, \quad \text{for every } y, z \in (D_0)_{r_0/3}, \end{aligned} \quad (3.15)$$

where C depends on A , L , λ , \bar{A} and n . Let $\rho_0 = \frac{r_0}{\bar{C}}$, where \bar{C} is the constant introduced in Theorem 3.4, let $r \in (0, d_2)$ and denote

$$w = P_k + \sigma \nu, \quad \text{where } \sigma = a^{\bar{h}-1} \lambda_1,$$

then

$$\tilde{S}_{\mathcal{U}_{k-1}}(\omega, \omega) = I_1(\omega) + I_2(\omega), \quad (3.16)$$

where

$$\begin{aligned} I_1(\omega) &= \int_{B_{\rho_0}(P_k) \cap D_k} (\gamma^{(1)} - \gamma^{(2)}) A(\cdot) \nabla \tilde{G}_1(\cdot, \omega) \cdot \nabla \tilde{G}_1(\cdot, \omega), \\ I_2(\omega) &= \int_{\mathcal{U}_{k-1} \setminus (B_{\rho_0}(P_k) \cap D_k)} (\gamma^{(1)} - \gamma^{(2)}) A(\cdot) \nabla \tilde{G}_1(\cdot, \omega) \cdot \nabla \tilde{G}_1(\cdot, \omega) \end{aligned}$$

and (see [A-V])

$$|I_2(\omega)| \leq CE\rho_0^{2-n}, \quad (3.17)$$

where C depends on λ , \bar{A} and n only. To estimate $I_1(\omega)$ we recall Theorem 3.4 which leads to

$$\begin{aligned} |I_1(\omega)| &\geq |\gamma_k^{(1)} - \gamma_k^{(2)}| C_1 \int_{B_{\rho_0}(P_k) \cap D_k} |\nabla_x \Gamma(Jx, J\omega)|^2 \\ &\quad - C_2 \int_{B_{\rho_0}(P_k) \cap D_k} |A(x)| |\nabla_x \Gamma(Jx, J\omega)| \frac{|x - \omega|^{1-n+\beta}}{\rho_0^\beta} \\ &\quad - C_3 \int_{B_{\rho_0}(P_k) \cap D_k} |A(x)| \frac{|x - \omega|^{2-2n+\beta}}{\rho_0^{2\beta}}, \end{aligned}$$

where C_1, C_2, C_3 are constants that depends on $M, \lambda, \alpha, \bar{A}$ and n only. Therefore, by combining (3.15) together with (3.16) and (3.17), we obtain

$$\begin{aligned} |I_1(\omega)| &\geq |\gamma_k^{(1)} - \gamma_k^{(2)}| C_1 \int_{B_{\rho_0}(P_k) \cap D_k} \frac{|J^2(x - \omega)|^2}{|J(x - \omega)|^{2n}} \\ &\quad - \frac{C_2 E}{\rho_0^\beta} \int_{B_{\rho_0}(P_k) \cap D_k} \frac{|J^2(x - \omega)|}{|J(x - \omega)|^n} |x - \omega|^{1-n+\beta} \\ &\quad - \frac{C_3 E}{\rho_0^{2\beta}} \int_{B_{\rho_0}(P_k) \cap D_k} |x - \omega|^{2(1-n)+\beta}. \end{aligned}$$

Therefore

$$|I_1(\omega)| \geq |\gamma_k^{(1)} - \gamma_k^{(2)}| C_1 \int_{B_{\rho_0}(P_k) \cap D_k} |x - \omega|^{2(1-n)} \quad (3.18)$$

$$- \frac{C_2 E}{\rho_0^\beta} \int_{B_{\rho_0}(P_k) \cap D_k} |x - \omega|^{2(1-n)+\beta} \quad (3.19)$$

$$- \frac{C_3 E}{\rho_0^{2\beta}} \int_{B_{\rho_0}(P_k) \cap D_k} |x - \omega|^{2(1-n)+\beta}, \quad (3.20)$$

which leads to

$$|I_1(\omega)| \geq C_1 |\gamma_k^{(1)} - \gamma_k^{(2)}| \sigma^{2-n} - C_2 E \frac{\sigma^{2-n+\beta}}{\rho_0^\beta}, \quad (3.21)$$

where β is the number introduced in Theorem 3.4 and C_1, C_2 depend on $M, \lambda, \alpha, \bar{A}$ and n only. By combining (3.21) together with (3.16) and (3.17) we obtain

$$C_1 |\gamma_k^{(1)} - \gamma_k^{(2)}| \sigma^{2-n} \leq |\tilde{S}_{\mathcal{U}_{k-1}}(\omega, \omega)| + C_2 E \frac{\sigma^{2-n+\beta}}{\rho_0^\beta} \quad (3.22)$$

and by Proposition 3.5 and (3.15) we obtain

$$|\tilde{S}_{\mathcal{U}_{k-1}}(\omega, \omega)| \leq \sigma^{2-n} C^{\bar{h}} (E + \varepsilon + \delta_{k-1}) \left(\omega^{\frac{1}{\bar{c}}} \left(\frac{\varepsilon + \delta_{k-1}}{E + \varepsilon + \delta_{k-1}} \right) \right)^{\left(\frac{1}{\bar{c}}\right)^{\bar{h}}},$$

where $C \geq 1$ is a constant depending on $A, L, \bar{A}, M, N, \alpha, \lambda$ and n only, therefore

$$|\gamma_k^{(1)} - \gamma_k^{(2)}| \leq C^{\bar{h}}(\varepsilon + \delta_{k-1} + E) \left(\omega_{1/C}^{(2(k-1))} \right)^{\left(\frac{1}{C}\right)^{\bar{h}}} + C_2 E \left(\frac{\sigma}{\rho_0} \right)^{\theta}. \quad (3.23)$$

We need to estimate $C^{\bar{h}}$ and $\left(\frac{1}{C}\right)^{\bar{h}}$, where $C > 1$. It turns out that

$$\begin{aligned} C^{\bar{h}} &\leq C^2 \left(\frac{d_1}{r} \right)^{-\frac{1}{\log_c a}} \\ \left(\frac{1}{C} \right)^{\bar{h}} &\leq \left(\frac{1}{C} \right)^2 \left(\frac{r}{d_1} \right)^{-\frac{1}{\log_c a}}, \end{aligned} \quad (3.24)$$

therefore

$$|\gamma_k^{(1)} - \gamma_k^{(2)}| \leq C(\varepsilon + \delta_{k-1} + E) \left(\left(\frac{d_1}{r} \right)^C \left(\omega_{1/C}^{(2(k-1))} \right)^{\left(\frac{r}{d_1}\right)^C} + \left(\frac{r}{d_1} \right)^{\theta} \right). \quad (3.25)$$

By (3.25) we obtain for every $k \in \{1, \dots, K\}$

$$\delta_k \leq \delta_{k-1} + C(\varepsilon + \delta_{k-1} + E) \left(\omega_{1/C}^{(2(k+1))} \left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right) \right)^{\frac{1}{C}},$$

which leads to

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)} \leq C(\varepsilon + E) \left(\omega_{\frac{1}{C}}^{(K^2)} \left(\frac{\varepsilon}{\varepsilon + E} \right) \right)^{\frac{1}{C}},$$

therefore

$$E \leq C(\varepsilon + E) \left(\omega_{\frac{1}{C}}^{(K^2)} \left(\frac{\varepsilon}{\varepsilon + E} \right) \right)^{\frac{1}{C}}. \quad (3.26)$$

Assuming that $E > \varepsilon e^2$ (if this is not the case then the theorem is proven) we obtain

$$E \leq C \left(\frac{E}{e^2} + E \right) \left(\omega_{\frac{1}{C}}^{(K^2)} \left(\frac{\varepsilon}{E} \right) \right)^{\frac{1}{C}},$$

which leads to

$$\frac{1}{C} \leq \omega_{\frac{1}{C}}^{(K^2)} \left(\frac{\varepsilon}{E} \right)$$

therefore

$$E \leq \frac{1}{\omega_{\frac{1}{C}}^{(-K^2)} \left(\frac{1}{C} \right)} \varepsilon,$$

which concludes the proof. □

4 Proof of technical propositions

4.1 Proof of the asymptotic estimates

Whenever φ is a Lipschitz continuous function on \mathbb{R}^{n-1} , we shall denote by $Q_{\varphi,r}^+$ and $Q_{\varphi,r}^-$ the following sets

$$Q_{\varphi,r}^+ = \{(x', x_n) \in Q_r \mid x_n > \varphi(x')\} , \quad (4.1)$$

$$Q_{\varphi,r}^- = \{(x', x_n) \in Q_r \mid x_n < \varphi(x')\} . \quad (4.2)$$

Let $0 < \mu < 1$ and $B^+ \in C^\mu(\overline{Q_{\varphi,r}^+})$, $B^- \in C^\mu(\overline{Q_{\varphi,r}^-})$ be symmetric, positive definite matrix valued functions and define

$$B(x) = \begin{cases} B^+(x), & x \in Q_{\varphi,r}^+, \\ B^-(x), & x \in Q_{\varphi,r}^-, \end{cases}$$

such that B satisfies the uniform ellipticity condition

$$\lambda_0^{-1}|\xi|^2 \leq B(x)\xi \cdot \xi \leq \lambda_0|\xi|^2 , \quad \text{for almost every } x \in Q_r, \\ \text{for every } \xi \in \mathbb{R}^n . \quad (4.3)$$

where $\lambda_0 > 0$ is a constant.

THEOREM 4.1. *Let $k > 0, r > 0$ and $0 < \alpha < 1$ be fixed numbers. Moreover, let B be a matrix as above. Assume that $\varphi \in C^{1,\alpha}(B'_r)$ and let $U \in H^1(Q_r)$ be a solution to*

$$\operatorname{div} \left(\left(1 + (k-1)\chi_{Q_{\varphi,r}^+} \right) B \nabla U \right) = 0 . \quad (4.4)$$

Suppose α' satisfies at the same time $0 < \alpha' \leq \mu$ and $\alpha' < \frac{\alpha}{(\alpha+1)n}$. Then, there exists a positive constant C such that for any $\rho \leq \frac{r}{2}$ and for any $x \in Q_{r-2\rho}$, the following estimate holds

$$\|\nabla U\|_{L^\infty(Q_\rho(x))} + \rho^{\alpha'} |\nabla U|_{\alpha', Q_\rho(x) \cap Q_{\varphi,r}^+} + \rho^{\alpha'} |\nabla U|_{\alpha', Q_\rho(x) \cap Q_{\varphi,r}^-} \\ \leq \frac{C}{\rho^{1+n/2}} \|U\|_{L^2(Q_{2\rho}(x))} , \quad (4.5)$$

where C depends on $\|\varphi\|_{C^{1,\alpha}(B'_r)}, k, \alpha, \alpha', n, \lambda_0, \|B^+\|_{C^{\alpha'}(\overline{Q_{\varphi,r}^+})}$ and $\|B^-\|_{C^{\alpha'}(\overline{Q_{\varphi,r}^-})}$ only.

Proof. For the proof we refer to [Li-Vo, Theorem 1.1], where the authors, among various results, obtain piecewise $C^{1,\alpha'}$ estimates for solutions to divergence form elliptic equations with piecewise Hölder continuous coefficients (see also [Li-Ni]). \square

We fix $l \in \{1, \dots, K-1\}$. There exists a rigid transformation of coordinates under which $P_{l+1} = 0$ and

$$\Sigma_l \cap Q_{\frac{r_0}{3}} = \{x \in Q_{\frac{r_0}{3}} \mid x_n = \varphi(x')\} , \quad (4.6)$$

$$D_{j_{l+1}} \cap Q_{\frac{r_0}{3}} = \{x \in Q_{\frac{r_0}{3}} \mid x_n > \varphi(x')\} , \quad (4.7)$$

$$D_{j_l} \cap Q_{\frac{r_0}{3}} = \{x \in Q_{\frac{r_0}{3}} \mid x_n < \varphi(x')\} , \quad (4.8)$$

where φ is a $C^{1,\alpha}$ function on $B'_{\frac{r_0}{3}}$ satisfying

$$\varphi(0) = |\nabla \varphi(0)| = 0, \quad \|\varphi\|_{C^{1,\alpha}(B'_{r_0})} \leq Mr_0. \quad (4.9)$$

Moreover, up to a possible replacement of γ with $\frac{\gamma}{\gamma_{j_l}}$, we can assume that $\gamma|_{D_{j_l}} = 1$ and $\gamma|_{D_{j_{l+1}}} = k$ where k is a real number which satisfies

$$\bar{\gamma} \leq k \leq \bar{\gamma}^{-2}. \quad (4.10)$$

Let τ be a C^∞ function on \mathbb{R} such that $0 \leq \tau \leq 1$, $\tau(s) = 1$ for every $s \in (-1, 1)$, $\tau(s) = 0$ for every $s \in \mathbb{R} \setminus (-2, 2)$ and $|\tau'(s)| \leq 2$ for every $s \in \mathbb{R}$.

We introduce

$$r_1 = \frac{r_0}{3} \min \left\{ \frac{1}{2} (8M)^{-\frac{1}{\alpha}}, \frac{1}{4} \right\} \quad (4.11)$$

and we consider the following change of variable $\xi = \Phi(x)$ given by

$$\begin{cases} \xi' = x', \\ \xi_n = x_n - \varphi(x') \tau\left(\frac{|x'|}{r_1}\right) \tau\left(\frac{x_n}{r_1}\right). \end{cases} \quad (4.12)$$

It can be verified that the map Φ is a $C^{1,\alpha}(\mathbb{R}^n, \mathbb{R}^n)$ and it satisfies the following properties

$$\Phi(\Sigma_l \cap Q_{r_1}) = \{x \in Q_{r_1} \mid x_n = 0\}, \quad (4.13)$$

$$\Phi(x) = x, \quad \text{for every } x \in \mathbb{R}^n \setminus Q_{2r_1}, \quad (4.14)$$

$$\begin{aligned} C^{-1}|x_1 - x_2| &\leq |\Phi(x_1) - \Phi(x_2)| \\ &\leq C|x_1 - x_2|, \quad \text{for every } x_1, x_2 \in \mathbb{R}^n, \end{aligned} \quad (4.15)$$

$$|\Phi(x) - x| \leq \frac{C}{r_0^\alpha} |x|^{1+\alpha}, \quad \text{and} \quad (4.16)$$

$$|D\Phi(x) - I| \leq \frac{C}{r_0^\alpha} |x|^\alpha, \quad \text{for every } x \in \mathbb{R}^n, \quad (4.17)$$

where $C, C > 1$, depends on M and α only and I denotes the identity matrix.

Let $y_n \in (-\frac{r_1}{2}, 0)$ and $y = ye_n$. We set

$$\eta = \Phi(y), \quad (4.18)$$

$$\tilde{G}(\xi, \eta) = G(\Phi^{-1}(\xi), \Phi^{-1}(\eta)), \quad (4.19)$$

$$J(\xi) = (D\Phi)(\Phi^{-1}(\xi)), \quad (4.20)$$

$$\tilde{\sigma}(\xi) = \frac{1}{\det J(\xi)} J(\xi) \gamma(\Phi^{-1}(\xi)) A(\Phi^{-1}(\xi)) (J(\xi))^T, \quad (4.21)$$

we have that $\tilde{G}(\cdot, \eta)$ is a solution to

$$\begin{cases} \operatorname{div}(\tilde{\sigma} \nabla(\xi) \tilde{G}(\cdot, \eta)) = -\delta(\cdot - \eta), & \text{in } \Omega, \\ \tilde{G}(\cdot, \eta) = 0, & \text{on } \partial\Omega. \end{cases} \quad (4.22)$$

We have

$$\tilde{\sigma}(\xi) = (1 + (k-1)\chi^+(\xi))B(\xi), \quad \text{for any } \xi \in Q_{r_1}, \quad (4.23)$$

where χ^+ is the characteristic function of \mathbb{R}_+^n and

$$B(\xi) = \frac{1}{\det J(\xi)} J(\xi) A(\Phi^{-1}(\xi)) (J(\xi))^T. \quad (4.24)$$

Furthermore, we have that B is of class C^α and

$$\|B\|_{C^{0,\alpha}(\Omega)} \leq C, \quad (4.25)$$

where $C > 0$ is a constant depending on $M, \alpha, \lambda, \bar{A}$ only. We also have that $B(0) = A(0)$. We denote

$$\sigma_0(\xi) = (1 + (k-1)\chi^+(\xi))A(\xi) \quad (4.26)$$

and with

$$\sigma_{0,0}(\xi) = (1 + (k-1)\chi^+(\xi))A(0) \quad (4.27)$$

and we refer to G_0 as to the Green function solution to

$$\begin{cases} \operatorname{div}(\sigma_{0,0}(\cdot) \nabla G_0(\cdot, y)) = -\delta(\cdot - y), & \text{in } \Omega, \\ G_0(\cdot, y) = 0, & \text{on } \partial\Omega. \end{cases} \quad (4.28)$$

We then define

$$R(\xi, \eta) = \tilde{G}(\xi, \eta) - G_0(\xi, \eta). \quad (4.29)$$

LEMMA 4.2. *For every $\xi \in B_{\frac{r_1}{4}}^+$ and $\eta_n \in (-\frac{r_1}{4}, 0)$ we have that*

$$|R(\xi, e_n \eta_n)| + |\xi - e_n \eta_n| |\nabla_\xi R(\xi, \eta)| \leq \frac{C}{r_1^\beta} |\xi - e_n \eta_n|^{\beta+2-n}, \quad (4.30)$$

where $\beta \in (0, \alpha^2]$ depends on α and n only and C depends on $M, \bar{\gamma}, \lambda, \bar{A}$ only.

Proof. It is easy to check that R in (4.29) satisfies

$$\begin{cases} \operatorname{div}_\xi(\tilde{\sigma}(\cdot) \nabla_\xi R(\cdot, \eta)) = -\operatorname{div}_\xi((\tilde{\sigma}(\cdot) - \sigma_{0,0}(\cdot)) \nabla_\xi G_0(\cdot, \eta)), & \text{in } \Omega, \\ R(\cdot, \eta) = 0, & \text{on } \partial\Omega. \end{cases} \quad (4.31)$$

By the representation formula over Ω we have that R in (4.29) satisfies

$$R(\xi, \eta) = \int_{\Omega} (\tilde{\sigma}(\zeta) - \sigma_{0,0}(\zeta)) \nabla_{\zeta} G_0(\zeta, \eta) \cdot \nabla \tilde{G}(\zeta, \xi) d\zeta . \quad (4.32)$$

We consider $\xi \in Q_{\frac{r_1}{2}}^+$ and $\eta = e_n \eta_n$ and we split R as the sum of the following integrals

$$R_1(\xi, \eta) = \int_{\Omega \setminus Q_{r_1}} (\tilde{\sigma}(\zeta) - \sigma_{0,0}(\zeta)) \nabla_{\zeta} G_0(\zeta, \eta) \cdot \nabla \tilde{G}(\zeta, \xi) d\zeta , \quad (4.33)$$

$$R_2(\xi, \eta) = \int_{Q_{r_1}} (\tilde{\sigma}(\zeta) - \sigma_{0,0}(\zeta)) \nabla_{\zeta} G_0(\zeta, \eta) \cdot \nabla \tilde{G}(\zeta, \xi) d\zeta . \quad (4.34)$$

By the bounds (2.8), (2.10), (2.9) and by combining the Schwartz inequality with the Caccioppoli inequality we get

$$|R_1(\xi, \eta)| \leq \frac{C}{r_1^2} \|G_0(\cdot, \eta)\|_{L^2(\Omega \setminus Q_{3r_1/4})} \|\tilde{G}(\cdot, \eta)\|_{L^2(\Omega \setminus Q_{3r_1/4})}, \quad (4.35)$$

where $C > 0$ depends on $M, \alpha, \bar{\gamma}, \lambda$ and \bar{A} only. By the standard behaviour (3.3) of the Green functions at hand, it follows that

$$|R_1(\xi, \eta)| \leq C r_1^{2-n} , \quad (4.36)$$

where $C > 0$ depends on $M, \alpha, \bar{\gamma}, \lambda$ and \bar{A} only. Moreover being $B(0) = A(0)$, it follows that (2.9) and (4.25) lead to

$$|\tilde{\sigma}(\xi) - \sigma_{0,0}(\xi)| \leq \max\{1, k\} (|B(\xi) - A(0)|) \leq \frac{C^\alpha}{r_1} |\xi|^\alpha, \quad (4.37)$$

for any $\xi \in Q_{r_1}$, where C depends on M, α, \bar{A} and $\bar{\gamma}$ only. Moreover, by (3.3) and by Theorem 4.1 we have that

$$|\nabla_{\zeta} G_0(\zeta, \xi)| \leq C |\zeta - \xi|^{1-n} , \quad \text{for every } \zeta, \xi \in Q_{r_1}, \quad (4.38)$$

where C depends on M, α, \bar{A} and $\bar{\gamma}$ only. By (4.15) and the same arguments used above, we infer that

$$|\nabla_{\zeta} \tilde{G}(\zeta, \xi)| \leq C |\zeta - \xi|^{1-n} , \quad \text{for every } \zeta, \xi \in Q_{r_1}, \quad (4.39)$$

where C depends on M, α, \bar{A} and $\bar{\gamma}$ only. We denote

$$I_1 = \int_{B_{4h}} |\zeta|^\alpha |\zeta - \xi|^{1-n} |\zeta - \eta|^{1-n} d\eta \quad (4.40)$$

and

$$I_2 = \int_{\mathbb{R}^n \setminus B_{4h}} |\zeta|^\alpha |\zeta - \xi|^{1-n} |\zeta - \eta|^{1-n} d\eta. \quad (4.41)$$

By (4.37), (4.38) and (4.39) we have that

$$|R_2(\xi, \eta)| \leq \frac{C}{r_1^\alpha} (I_1 + I_2) . \quad (4.42)$$

Let us denote now $h = |\xi - \eta|$ and consider the following change of variables $\zeta = hw$; we set $t = \frac{\xi}{h}$ and $s = \frac{\eta}{h}$ and it follows that for any $t, s \in \mathbb{R}^n$ we have that $|t - s| = 1$. We obtain that

$$I_1 \leq 4^\alpha h^{\alpha+2-n} \int_{B_4} |t - w|^{1-n} |s - w|^{1-n} dw . \quad (4.43)$$

Let us now set

$$F(t, s) = \int_{B_4} |t - w|^{1-n} |s - w|^{1-n} dw . \quad (4.44)$$

From standard bounds (see for instance, [Mi, Chapter 2]) we have that

$$F(t, s) \leq C, \quad (4.45)$$

where C depends on n only. Hence

$$I_1 \leq Ch^{\alpha+2-n}, \quad (4.46)$$

where C depends on n only. We consider now integral I_2 . We recall that $\eta = e_n \eta_n$, where $\eta_n \in (-\frac{r_1}{2}, 0)$ and $\xi \in Q_{\frac{r_1}{2}}^+$, hence we have

$$|\eta| = -\eta_n \leq -\eta_n + \xi_n \leq |\xi - \eta| = h, \quad (4.47)$$

which leads to

$$|\xi| \leq |\xi - \eta| + |\eta| \leq 2h . \quad (4.48)$$

On the other hand, we have that for any $\zeta \in \mathbb{R}^n \setminus B_{4h}$

$$|\zeta| \leq |\zeta - \eta| + |\eta| \leq |\zeta - \eta| + \frac{1}{4}|\zeta|, \quad (4.49)$$

hence we get

$$\frac{3}{4}|\zeta| \leq |\zeta - \eta|. \quad (4.50)$$

and by using the same arguments we get

$$\frac{1}{2}|\zeta| \leq |\xi - \zeta|, \text{ for any } \zeta \in \mathbb{R}^n \setminus B_{4h}. \quad (4.51)$$

By combining (4.50) together with (4.51), we obtain that

$$I_2 \leq \left(\frac{8}{3}\right)^{1-n} \int_{\mathbb{R}^n \setminus B_{4h}} |\zeta|^{\alpha+2-2n} d\zeta \leq Ch^{\alpha+2-n}, \quad (4.52)$$

where C depends on α and n only. By combining (4.36), (4.42), (4.46) and (4.52) we obtain

$$|R(\xi, \eta)| \leq \frac{C}{r_1^\alpha} h^{\alpha+2-n}, \quad (4.53)$$

where C depends on M, α, \bar{A}, n and $\bar{\gamma}$ only. Let us fix $\xi \in B_{\frac{r_1}{4}}^+$ and $\eta_n \in (-r_1/4, 0)$ and consider the cylinder

$$Q = B'_{\frac{h}{8}}(\xi') \times \left(\xi_n, \xi_n + \frac{h}{8}\right). \quad (4.54)$$

Observing that $h = |\xi - (0, \eta_n e_n)| \leq \frac{r_1}{2}$ we deduce that $Q \subset Q_{\frac{r_1}{2}}^+$. Moreover $Q \subset Q_{\frac{h}{4}(\xi)}$ and $\xi \in \partial Q$, then by choosing for instance $\alpha' = \frac{1}{2} \min \left\{ \alpha, \frac{\alpha}{(\alpha+1)n} \right\}$ in the statement of Theorem 4.1 and observing that $(0, \eta_n e_n) \notin Q_{\frac{h}{2}}(\xi)$, by (4.5) we obtain the following bound for the seminorm

$$\begin{aligned} |\nabla_\xi \tilde{G}(\cdot, e_n \eta_n)|_{\alpha', Q} &\leq |\nabla_\xi \tilde{G}(\cdot, e_n \eta_n)|_{\alpha', Q_{\frac{h}{4}(\xi)} \cap Q_{\frac{r_1}{2}}^+} \\ &\leq Ch^{-\alpha'-1-n/2} \|\nabla_\xi \tilde{G}(\cdot, e_n \eta_n)\|_{L^2(Q_{\frac{h}{2}(\xi)})}, \end{aligned} \quad (4.55)$$

where C depends on M, α, \bar{A}, n and $\bar{\gamma}$ only. Furthermore by observing that for any $\tilde{\xi} \in Q_{\frac{h}{2}(\xi)}$ we have that $|\tilde{\xi} - (0, e_n \eta_n)| \geq \frac{h}{2}$ and by (3.3) we have that

$$|\nabla_\xi \tilde{G}(\cdot, e_n \eta_n)|_{\alpha', Q} \leq Ch^{\alpha'+1-n}, \quad (4.56)$$

where C depends on M, α, \bar{A}, n and $\bar{\gamma}$ only. By analogous argument we may also infer that

$$|\nabla_\xi G_0(\cdot, e_n \eta_n)|_{\alpha', Q} \leq Ch^{\alpha'+1-n}, \quad (4.57)$$

where C depends on M, α, \bar{A}, n and $\bar{\gamma}$ only. Hence by (4.29), (4.56) and (4.57) we obtain

$$|\nabla_\xi R(\cdot, e_n \eta_n)|_{\alpha', Q} \leq Ch^{\alpha'+1-n}, \quad (4.58)$$

where C depends on M, α, \bar{A}, n and $\bar{\gamma}$ only. We recall the following interpolation inequality (see for instance [A-S, Proposition 8.3])

$$\|\nabla_\xi R(\cdot, e_n \eta_n)\|_{L^\infty(Q)} \leq \|R(\cdot, e_n \eta_n)\|_{L^\infty(Q)}^{\frac{\alpha'}{1+\alpha'}} |\nabla_\xi R(\cdot, e_n \eta_n)|_{\alpha', Q}^{\frac{1}{1+\alpha'}}, \quad (4.59)$$

where C depends on M, α, \bar{A}, n and $\bar{\gamma}$ only. By the above estimate and (4.53) we get

$$|\nabla_\xi R(\xi, e_n \eta_n)| \leq \frac{C}{r_1^\beta} h^{\beta+1-n}, \text{ for every } \xi \in B_{\frac{r_1}{4}}^+ \text{ and } \eta \in \left(-\frac{r_1}{4}, 0\right), \quad (4.60)$$

where C depends on M, α, \bar{A}, n and $\bar{\gamma}$ only. The thesis follows with $\beta = \frac{\alpha'^2}{1+\alpha'}$. \square

Proof of Theorem 3.4. We first assume that the auxiliary hypothesis that $A(0) = I$ is fulfilled and denote with $H(\xi, \eta)$ the half space fundamental solution of the operator $\operatorname{div}_\xi((1 + (k-1))\chi^+(\xi)I(\xi)\nabla_\xi)$ which has the following explicit form

$$H(\xi, \eta) = \begin{cases} \frac{1}{k}\Gamma(\xi, \eta) + \frac{k-1}{k(k+1)}\Gamma(\xi, \eta^*) , & \text{if } \xi_n, \eta_n > 0 \\ \frac{2}{k+1}\Gamma(\xi, \eta) , & \text{if } \xi_n \cdot \eta_n < 0 \\ \Gamma(\xi, \eta) + \frac{1-k}{k+1}\Gamma(\xi, \eta^*) , & \text{if } \xi_n, \eta_n < 0 \end{cases} \quad (4.61)$$

where Γ is the distribution introduced in (3.1) and for any $\xi = (\xi', \xi_n)$ we denote $\xi^* = (\xi', -\xi_n)$. Let $\eta_n \in (-\frac{r_1}{4}, 0)$, then we have that

$$\begin{cases} \operatorname{div}_\xi((1 + (k-1))\chi^+(\xi)I(\xi)\nabla_\xi(G_0(\xi, e_n\eta_n) - H(\xi, e_n\eta_n))) = 0 , & \text{in } Q_{\frac{r_1}{2}} , \\ |(G_0(\xi, e_n\eta_n) - H(\xi, e_n\eta_n))| \leq Cr_1^{n-2}, & \text{for any } \xi \in \partial Q_{\frac{r_1}{2}} . \end{cases}$$

Hence by the maximum principle we can infer that

$$\|G_0(\cdot, e_n\eta_n) - H(\cdot, e_n\eta_n)\|_{L^\infty(Q_{\frac{r_1}{2}})} \leq Cr_1^{n-2} \quad (4.62)$$

and by Theorem 4.1 we deduce that

$$\|\nabla_\xi G_0(\cdot, e_n\eta_n) - \nabla_\xi H(\cdot, e_n\eta_n)\|_{L^\infty(Q_{\frac{r_1}{4}})} \leq Cr_1^{n-1} . \quad (4.63)$$

We now consider a point $x \in \Phi^{-1}(B_{\frac{r_1}{4}}^+)$ and $y_n \in (-\frac{r_1}{2}, 0)$, then we observe that being $\Phi(y) = y$ we have that

$$|\Phi(y)| = |\Phi(y) - \Phi(0)| \leq |\Phi(y) - \Phi(x)| . \quad (4.64)$$

Moreover, by (4.15) and the above estimate we have that

$$C^{-1}|x| \leq |\Phi(x)| \leq |\Phi(x) - \Phi(y)| + |\Phi(y)| \leq C|x - y| . \quad (4.65)$$

By combining the above estimate with (4.16), we infer that

$$|\Phi(x) - x| \leq \frac{C}{r_0^\alpha}|x|^{1+\alpha} \leq \frac{C}{r_0^\alpha}|x - e_n y_n|^{1+\alpha} , \quad (4.66)$$

where C depends on M and α only. Let $\{A_k\}_{k \geq 1}$ be a regularizing sequence for A obtained by convolution with a sequence of mollifiers, then we have that

$$\|A_k\|_{C^1(\Omega)} \leq 2\bar{A}, \quad \text{for any } k \in \mathbb{N} \quad (4.67)$$

and A_k satisfies (2.10), with $A = A_k$, $k \in \mathbb{N}$. Let us introduce the following function

$$F_k : B_{r_0} \setminus \{e_n y_n\} \rightarrow \mathbb{R} \quad (4.68)$$

$$z \mapsto \langle A_k(z)(z - e_n y_n), (z - e_n y_n) \rangle^{\frac{2-n}{2}}, \quad (4.69)$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product of vectors in \mathbb{R}^n . Given $z_1, z_2 \in B_{r_0} \setminus \{e_n y_n\}$ by the Mean-Value Theorem, there exists $t_k, 0 < t_k < 1$ such that

$$\begin{aligned} |F_k(z_1) - F_k(z_2)| &\leq C|z_1 - z_2| \left(\left| \langle A_k(z_{t_k})(z_{t_k} - e_n y_n), (z_{t_k} - e_n y_n) \rangle^{\frac{1-n}{2}} \right| \right. \\ &\quad \left. + \left| \langle A_k(z_{t_k})(z_{t_k} - e_n y_n), (z_{t_k} - e_n y_n) \rangle^{-\frac{n}{2}} \right| \right. \\ &\quad \left. \times \left| \langle \sum_{i=1}^n \partial_{z_i} A_k(z_{t_k})(z_{t_k} - e_n y_n), (z_{t_k} - e_n y_n) \rangle \right| \right) \end{aligned}$$

where $z_{t_k} = z_1 + t_k(z_2 - z_1)$ and where C depends on M, α, \bar{A} and n only. Let us denote with Γ_k the fundamental solution introduced in (3.9) associated to the matrix A_k . We choose $z_1 = \Phi(x)$ and $z_2 = x$ and we have that

$$|\Gamma_k(\Phi(x), e_n y_n) - \Gamma_k(x, e_n y_n)| \leq C|\Phi(x) - x||x - e_n y_n + t_k(\Phi(x) - x)|^{1-n},$$

C depends on $M, \alpha, \bar{A}, \lambda$ and n only. By (4.66) and the triangle inequality we deduce that for any $x \in D_{j_{i+1}} \cap B_{\frac{r_0}{4C^{1/\alpha}}}$ we get

$$|x - e_n y_n - t_k(\Phi(x) - x)| \geq |x - e_n y_n| - |t_k||\Phi(x) - x| \quad (4.70)$$

$$\geq |x - e_n y_n| - |x - e_n y_n|^{1+\alpha} \geq \frac{1}{2}|x - e_n y_n|. \quad (4.71)$$

Finally combining the above estimates and (4.66) we obtain

$$|\Gamma_k(\Phi(x), e_n y_n) - \Gamma_k(x, e_n y_n)| \leq C|x - e_n y_n|^{2-n+\alpha}, \quad (4.72)$$

where C depends on $M, \alpha, \lambda, \bar{A}$ and n only. Now since A_k converges uniformly to A in $\bar{\Omega}$ we can infer that

$$|\Gamma(\Phi(x), e_n y_n) - \Gamma(x, e_n y_n)| \leq C|x - e_n y_n|^{2-n+\alpha}, \quad (4.73)$$

for $x \in \Phi^{-1}(B_{\frac{r_1}{4}}^+)$, where C depends on $M, \alpha, \lambda, \bar{A}$ and n only. By (4.62), (4.63) and (4.73) we have

$$\begin{aligned} |G_0(\Phi(x), e_n y_n) - H(x, e_n y_n)| &\leq |G_0(\Phi(x), e_n y_n) - H(\Phi(x), e_n y_n)| \\ &\quad + |H(\Phi(x), e_n y_n) - H(x, e_n y_n)| \\ &\leq \frac{C}{r_0^\alpha} |x - e_n y_n|^{\alpha+2-n} \end{aligned} \quad (4.74)$$

and

$$|\nabla G_0(\Phi(x), e_n y_n) - \nabla H(x, e_n y_n)| \leq \frac{C}{r_0^\alpha} |x - e_n y_n|^{\alpha+1-n}, \quad (4.75)$$

for $x \in \Phi^{-1}(B_{\frac{r_1}{4}}^+)$, where C depends on $M, \lambda, \bar{\gamma}, \alpha$ and n only. Moreover, by Lemma 4.2, (4.15) and recalling that $\Phi(y) = y$, we get

$$|R(\Phi(x), e_n \eta_n)| + |x - e_n \eta_n| |\nabla_\xi R(\Phi(x), \eta)| \leq \frac{C}{r_1^\beta} |\xi - e_n \eta_n|^{\beta+2-n}, \quad (4.76)$$

for $x \in \Phi^{-1}(B_{\frac{r_1}{4}}^+)$ and where C depends on $M, \lambda, \bar{\gamma}, \alpha$ and n only. Gathering (4.74), (4.75), (4.76) and recalling that

$$G(\bar{x}, e_n y_n) = G_0(\Phi(\bar{x}), e_n y_n) + R(\Phi(\bar{x}), e_n y_n) \quad (4.77)$$

we first find that

$$\left| G(\bar{x}, e_n y_n) - \frac{1}{1+k} \Gamma(\bar{x}, e_n y_n) \right| \leq \frac{C}{r_0^\beta} |\bar{x} - e_n y_n|^{\beta+2-n}, \quad (4.78)$$

$$\left| \nabla_x G(\bar{x}, e_n y_n) - \frac{1}{1+k} \nabla_x \Gamma(\bar{x}, e_n y_n) \right| \leq \frac{C}{r_0^\beta} |\bar{x} - e_n y_n|^{\beta+1-n}, \quad (4.79)$$

for a.e. $\bar{x} \in D_{j_{l+1}} \cap B_{\frac{r_0}{(4C)^{1/\alpha}}}$ and $y_n \in (-r_1/(4C)^{1/\alpha}, 0)$, where C depends on $M, \lambda, \bar{\gamma}, \bar{A}, \alpha$ and n only. The thesis then follows for the case $A(0) = I$.

To treat the general case when $A(0) \neq I$, we introduce the fundamental solution $H_{A(0)}$ of the operator $\operatorname{div}_\xi((1+(k-1))\chi^+(\xi)A(0)\nabla_\xi)$. We set $\sigma_I(\xi) = (1+(k-1))\chi^+(\xi)Id$ and $\sigma_{A(0)}(\xi) = (1+(k-1))\chi^+(\xi)A(0)$. Let us introduce the linear change of variable

$$L : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (4.80)$$

$$\xi \mapsto x = L\xi := R\sqrt{A^{-1}(0)}\xi, \quad (4.81)$$

where R is the planar rotation in \mathbb{R}^n that rotates the unit vector $\frac{v}{\|v\|}$, where $v = \sqrt{A(0)}e_n$, to the n -th standard unit vector e_n and such that

$$R|_{(\pi)^\perp} \equiv Id|_{(\pi)^\perp},$$

where π is the plane in \mathbb{R}^n generated by e_n, v and $(\pi)^\perp$ denotes the orthogonal complement of π in \mathbb{R}^n . For this choice of L we have

$$\text{i)} \quad A(0) = L^{-1} \cdot (L^{-1})^T,$$

$$\text{ii)} \quad (L\xi) \cdot e_n = \frac{1}{\|v\|} \xi \cdot v.$$

which leads to

$$\sigma_{A(0)}(\xi) = L^{-1} \sigma_I(L\xi) (L^{-1})^T,$$

which means that $L^{-1} : x \mapsto \xi$ is the linear change of variables that maps $\sigma_I(x)$ into $\sigma_{A(0)}(\xi)$. Therefore the fundamental solution for the operator $\operatorname{div}_\xi((1+(k-1))\chi^+(\xi)A(0)\nabla_\xi)$ turns out to be

$$H_{A(0)}(\xi, \eta) = \begin{cases} \sqrt{\det A^{-1}(0)} \left(\frac{1}{k} \Gamma(L\xi, L\eta) + \frac{k-1}{k(k+1)} \Gamma(L\xi, L^*\eta) \right), & \text{if } \xi_n, \eta_n > 0 \\ \sqrt{\det A^{-1}(0)} \left(\frac{2}{k+1} \Gamma(L\xi, L\eta) \right), & \text{if } \xi_n \cdot \eta_n < 0 \\ \sqrt{\det A^{-1}(0)} \left(\Gamma(L\xi, L\eta) + \frac{1-k}{k+1} \Gamma(L\xi, L^*\eta) \right), & \text{if } \xi_n, \eta_n < 0 \end{cases} \quad (4.82)$$

where the matrix $L^* = \{l_{i,j}^*\}_{i,j=1}^n$ is such that $l_{i,j}^* = l_{i,j}$ for $i = 1, \dots, n-1, j = 1, \dots, n$ and $l_{n,j}^* = -l_{n,j}$ for $j = 1, \dots, n$. In particular we have that when $\xi_n \cdot \eta_n < 0$ then

$$H_{A(0)}(\xi, \eta) = \sqrt{\det A^{-1}(0)} \frac{2}{k+1} < A^{-1}(0)(\xi - \eta, \xi - \eta) >^{\frac{2-n}{2}}.$$

Hence for the case $A(0) \neq I$ (4.78) and (4.79) shall be replaced by

$$\left| G(\bar{x}, e_n y_n) - \frac{1}{1+k} < A^{-1}(0)(\bar{x} - e_n y_n, \bar{x} - e_n y_n) >^{\frac{2-n}{2}} \right| \leq \frac{C}{r_0^\beta} |\bar{x} - e_n y_n|^{\beta+2-n},$$

$$\left| \nabla_x G(\bar{x}, e_n y_n) - \frac{1}{1+k} \nabla_x < A^{-1}(0)(\bar{x} - e_n y_n, \bar{x} - e_n y_n) >^{\frac{2-n}{2}} \right| \leq \frac{C}{r_0^\beta} |\bar{x} - e_n y_n|^{\beta+1-n},$$

for $\bar{x} \in D_{j_{l+1}} \cap B_{\frac{r_0}{(4C)^{1/\alpha}}}$ and $y_n \in (-r_1/(4C)^{1/\alpha}, 0)$ where C depends on $M, \lambda, \bar{\gamma}, \bar{A}, \alpha$ and n only. Hence the thesis follows also for the general case. \square

4.2 Proof of unique continuation estimates

Let $P_1, D_0, \Omega_0, (D_0)_r$ and \tilde{G}_i , for $i = 1, 2$ be as in subsection 3.2.1. Let us fix $k \in \{2, \dots, N\}$ and recall that there exist $j_1, \dots, j_K \in \{2, \dots, N\}$ such that

$$D_{j_1} = D_1, \dots, D_{j_K} = D_K.$$

We recall that

$$\mathcal{W}_K = \bigcup_{i=0}^K D_{j_i}, \quad \mathcal{U}_k = \Omega_0 \setminus \overline{\mathcal{W}_K}, \quad \text{when } k \geq 0$$

$(D_{j_0} = D_0)$ and for any $y, z \in \mathcal{W}_K$

$$\tilde{S}_{\mathcal{U}_K}(y, z) = \int_{\mathcal{U}_K} (\tilde{\sigma}_A^{(1)} - \tilde{\sigma}_A^{(2)}) \nabla \tilde{G}_1(\cdot, y) \cdot \nabla \tilde{G}_2(\cdot, z), \quad \text{when } k \geq 0.$$

The proof of Proposition 3.5 is a straight forward consequence of the following result (see [A-V][proof of Proposition 4.6]).

Proposition 4.3. *Let v be a weak solution to*

$$\operatorname{div}(\tilde{\sigma} \nabla v) = 0, \quad \text{in } \mathcal{W}_k,$$

where $\tilde{\sigma}$ is either equal to $\tilde{\sigma}_A^{(1)}$ or to $\tilde{\sigma}_A^{(2)}$. Assume that, for given positive numbers ε_0 and E_0 , v satisfies

$$|v(x)| \leq \varepsilon_0 r_0^{2-n}, \quad \text{for every } x \in (D_0)_{\frac{r_0}{3}}, \quad (4.1)$$

and

$$|v(x)| \leq E_0 (r_0 d(x))^{1-n/2}, \quad \text{for every } x \in \mathcal{W}_k, \quad (4.2)$$

where $d(x) = \text{dist}(x, \Sigma_{k+1})$. Then the following inequality holds true for every $r \in (0, d_1]$

$$|v(w_{\bar{h}}(P_{k+1}))| \leq r_0^{2-n} C^{\bar{h}} (E_0 + \varepsilon_0) \left(\omega_{1/C}^{(k)} \left(\frac{\varepsilon_0}{E_0 + \varepsilon_0} \right) \right)^{(1/C)^{\bar{h}}}. \quad (4.3)$$

Proof. We observe that the proof of this result follows the same line of the argument used in [A-V][proof of Proposition 4.4] which is independent from the presence of isotropy/anisotropy in $\tilde{\sigma}$. In fact their proof is based on an argument of unique continuation which require $\tilde{\sigma}$ to be Lipschitz continuous and the interfaces between each domain D_j to contain a $C^{1,\alpha}$ portion, therefore we simply recall [A-V][proof of Proposition 4.4] for a complete proof of this proposition. Here we simply recall for sake of completeness the main fact proven in [A-V][proof of Proposition 4.4]. By defining the quantities

$$r_1 = \frac{r_0}{4}, \quad \bar{\rho} = \frac{r_1}{128\sqrt{1+L^2}}$$

let $y_m \in D_m$ be a point "near the portion" Σ_{m+1} of the interface between D_m and D_{m+1} defined by

$$y_m = P_{m+1} - \frac{r_1}{32} \nu(P_{m+1}),$$

where $P_{m+1} \in \Sigma_{m+1}$. Their main point is the proof of the following fact

$$\|v\|_{L^\infty(B_{\bar{\rho}}(y_m))} \leq r_0^{2-n} C^{m+1} (E_0 + \varepsilon_0) \omega_{\frac{1}{C}}^{(m+1)} \left(\frac{\varepsilon_0}{E_0 + \varepsilon_0} \right), \quad (4.4)$$

where $\bar{\rho}$ has been chosen above so that $B_{\bar{\rho}}(y_m) \subset D_m$. The proof of the above inequality is done by induction. A so-called argument of *global propagation of smallness* is used there to prove (4.4) for $m = 0$. We refer to [A-R-R-V], Theorem 5.3 for a complete treatment of this topic. The rest of the proof is based on the *three sphere inequality*, therefore we simply refer to [A-V][proof of Proposition 4.4] for this. \square

Acknowledgments

The authors gratefully acknowledge the fruitful conversations with G. Alessandrini who kindly exchanged ideas about the global stability issue with the authors during the preparation of this work.

References

- [A] Alessandrini G. (1988) Stable determination of conductivity by boundary measurements, *App. Anal.* 27: 153-172.
- [A1] Alessandrini G. (1990) Singular Solutions of Elliptic Equations and the Determination of Conductivity by Boundary Measurements, *J. Differential Equations* 84 (2): 252-272.
- [A2] Alessandrini G. (1991) Determining conductivity by boundary measurements, the stability issue, *Applied and Industrial Mathematics, R. Spigler (ed.), Kluwer*: 317-324.
- [A3] Alessandrini G. (2007) Open issues of stability for the inverse conductivity problem, *J. Inv. Ill-Posed Problems* 15: 1-10.

- [A-B-R-V] Alessandrini G., Beretta E., Rosset E., Vessella S. (2000) Optimal stability for inverse elliptic boundary value problems with unknown boundaries, *Ann. Scuola Norm. Sup. Pisa, Cl. Sci. XXXIX* 29 (4): 755-806.
- [A-R-R-V] Alessandrini G., Rondi L., Rosset E. and Vessella S. (2009) The stability for the Cauchy problem for elliptic equations (topical review), *Inverse Problems* 25 (123004): 47pp.
- [A-G] G. Alessandrini and Gaburro R. (2001) Determining Conductivity with Special Anisotropy by Boundary Measurements, *SIAM J. Math. Anal.* 33: 153-171.
- [A-G1] Alessandrini G. and Gaburro R. (2009) The local Calderón problem and the determination at the boundary of the conductivity, *Comm. Partial Differential Equations* 34: 918-936.
- [A-S] Alessandrini G. and Sincich E., Cracks with impedance, stable determination from boundary data (2013) *Indiana Univ. Math.J.* 62 (3) :947-989.
- [A-V] Alessandrini G. and Vessella S. (2005) Lipschitz stability for the inverse conductivity problem, *Advances in Applied Mathematics* 35: 207-241.
- [A-L-P] Astala K. ,Lassas M. and Päiväranta L. (2005) Calderón inverse problem for anisotropic conductivity in the plane, *Comm. Partial Differential Equations* 30: 207-224.
- [B-B-R] Barceó J. A. , Barceó T. and Ruiz A. (2001) Stability of the inverse conductivity problem in the plane for less regular conductivities, *J. Differential Equations* 173: 231-270.
- [B-F-R] Barceó T. , Faraco D. and Ruiz A. (2007) Stability of Calderón inverse conductivity problem in the Plane, *Journal de Mathématiques Pures et Appliquées* 88 (6): 522-556.
- [Be] Belishev M. I. (2003) The Calderón Problem for Two-Dimensional Manifolds by the BC-Method, *SIAM J. Math. Anal.* 35 (1): 172182.
- [Be-dH-Q] Beretta E. , De Hoop M. , Qiu L. (2013) Lipschitz stability of an inverse boundary value problem for a Schrödinger type equation, *SIAM J. Math. Anal.* In print.
- [Be-Fr] Beretta E. and Francini E. (2011) Lipschitz stability for the electrical impedance tomography problem: the complex case, *Comm. Partial Differential Equations* 36: 1723-1749.
- [Be-Fr-V] Beretta E. and Francini E. and Vessella S., Uniqueness and Lipschitz stability for the identification of Lamé parameters from boundary measurements, *preprint* (link to arXiv <http://arxiv.org/abs/1303.2443>).
- [Bo] Borcea L. (2002) Electrical impedance tomography, *Inverse Problems* 18: R99-R136.
- [C] Calderón A. P. (1980) On an inverse boundary value problem, *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro)*: 65-73, Soc. Brasil. Mat., Rio de Janeiro. (Reprinted in 2006) *Comput. Appl. Math.* 25 (2-3): 133-138.
- [Ch-I-N] Cheney M., Isaacson D. and Newell J. C. (1999) Electrical impedance tomography, *SIAM Rev.* 41 (1): 85-101.
- [F-K-R] Faraco D. , Kurylev Y. and Ruiz A. (2013) G-convergence, Dirichlet to Neumann maps and invisibility, link to arXiv <http://arxiv-web3.library.cornell.edu/abs/1311.5466>
- [G-L] Gaburro R. and Lionheart W. R. B. (2009) Recovering Riemannian metrics in monotone families from boundary data, *Inverse Problems* 25 (4): 045004.

- [K-Vo1] Kohn R. and Vogelius M. (1984) Identification of an Unknown Conductivity by Means of Measurements at the Boundary, *SIAM-AMS Proc.* 14: 113-123.
- [K-Vo2] Kohn R. and Vogelius M. (1985) Determining Conductivity by Boundary Measurements II. Interior Results, *Comm. Pure Appl. Math.* 38: 643-667.
- [L] Lionheart W. R. B. (1997) Conformal Uniqueness Results in Anisotropic Electrical Impedance Imaging, *Inverse Problems* 13: 125-134.
- [La-U] Lassas M. and Uhlmann G. (2001) On determining a Riemannian manifold from the Dirichlet-to-Neumann map, *Ann. Sci. École Norm. Sup.* (4) 34 , (5): 771-787.
- [Liu] Liu L. (1997) Stability estimates for the two-dimensional inverse conductivity problem, PhD Thesis, University of Rochester, New York.
- [Lit-St-W] Littman W., Stampacchia G. and Weinberger H.W. (1963) Regular points for elliptic equations with discontinuous coefficients, *Ann. Scuola Norm. Pisa Cl. Sci.* 3 (17): 43-77.
- [Li-Ni] Li Y.Y. and Nirenberg L. (2003) Estimates for elliptic systems from composite material, *Comm. Pure Appl. Math.* LVI: 892-925.
- [Li-Vo] Li Y.Y. and Vogelius M. (2000) Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients, *Arch. Rational Mech. Anal.* 153: 91-151.
- [Ma] Mandache N. (2001) Exponential instability in an inverse problem for the Schrödinger equation, *Inverse Problems* 17: 1435-1444.
- [Mi] Miranda C. (1970) Partial differential equations of elliptic type, second ed., Springer, Berlin.
- [Na] Nachman A. (1995) Global Uniqueness for a two Dimensional Inverse Boundary Value Problem, *Ann. of Math.* 142:71-96.
- [S-U] Sylvester J. and Uhlmann G. (1987) A Global Uniqueness Theorem for an Inverse Boundary Valued Problem, *Ann. of Math.* 125: 153-169.
- [T] Trytten G.N. (1963) Pointwise bounds for solutions of the Cauchy problem for elliptic equations, *Arch. Rational Mech. Anal.* 13: 222-244.
- [U] Uhlmann G. (2009) Electrical impedance tomography and Calderón's problem (topical review), *Inverse Problems* 25 (12): 123011 doi:10.1088/0266-5611/25/12/123011.